Lisans araştırma dersleri

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Makale Çalışmaları

Reduction of topologically massive gravity I: Hamiltonian analysis of second order degenerate Lagrangians (29.03.2020)

1 Hamiltonian Analysis of The Second Order Lagrangians

1.1 Jacobi-Ostrogradsky momenta

$L[X] = L(X, \dot{X}, \ddot{X})$ 2. dereceden bir Langrangian density fonksiyonudur. Fonksiyonel diferansiyeli alındığında Euler lagrenge ve bir tam türevin toplamı olarak yazılabilir. İlk önce ikinci dereceden Lagrangian için Euler-Lagrange denklemini çıkaralım

$$0 = \delta S = \int_{t_a}^{t_b} dt\ [L(q + \delta q, \dot{q} + \delta\dot{q}, \ddot{q} + \delta\ddot{q}, t) - L(q, \dot{q}, \ddot{q}, t)] \tag{1}$$

$$0 = \int_{t_a}^{t_b} dt\ (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}) \tag{2}$$

$\delta\dot{q} = \delta(\frac{d}{dt}q) = \frac{d}{dt}(\delta q)$ ve $\delta\ddot{q} = \delta\frac{d}{dt}\dot{q} = \frac{d}{dt}(\delta\dot{q})$ eşitliklerini kullanıp 2. ve 3. terim için integration by parts kullanalım.

$$0 = \int_{t_a}^{t_b} dt\ (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta\dot{q}) \tag{3}$$

2. integralde $u = \frac{\partial L}{\partial \dot{q}}$, ise $du = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})dt$ ve $v = \delta q$ ise $dv = \frac{d}{dt}(\delta q)dt$ integration by parts kullanırsak. İntegral dışına çıkan ifadeler uçlarda varyasyon 0 oluğundan yok olur.

$$\int_{t_a}^{t_b} dt\ \frac{\partial L}{\partial q}\delta q \ \ + \ \ \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_a}^{t_b} \ -\int_{t_a}^{t_b} \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\ dt\ \delta q\ +\int_{t_a}^{t_b} dt\frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta \dot{q} = 0 \tag{4}$$

$$\int_{t_a}^{t_b} dt\delta q(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}) + \int_{t_a}^{t_b} dt\frac{\partial L}{\partial \ddot{q}}\frac{d}{dt}\delta \dot{q} = 0 \tag{5}$$

Sağdaki integral için: $u = \frac{\partial L}{\partial \ddot{q}}$ ise $du = dt\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$ ve $v = \delta \dot{q}$ ise $dv = dt\frac{d}{dt}\delta \dot{q}$ (uçlarda varyasyon 0 olduğundan integral dışına çıkan kısmı yazmıyorum.)

$$\int_{t_a}^{t_b} dt\delta q(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}) - \int_{t_a}^{t_b} dt\delta \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} = 0 \tag{6}$$

Sağdaki integral için tekrar integration by parts yaparsak. $u = \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$ ise $du = dt\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}$ ve $v = \delta q$ iken $dv = dt\frac{d}{dt}\delta q$

$$\int_{t_a}^{t_b} dt\delta q(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}) + \int_{t_a}^{t_b} dt\delta q\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \tag{7}$$

$$\int_{t_a}^{t_b} dt\delta q(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}) = 0 \tag{8}$$

Böylece Euler- Lagrange denklemi elde edilir.

$$\varepsilon_x(L[X]) = \frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} = 0 \tag{9}$$

$$d(L[X]dt) = \frac{\partial L}{\partial X}dX + \frac{\partial L}{\partial \dot{X}}d\dot{X} + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} = \varepsilon_x(L[X]).dX + \frac{d}{dt}\theta_L[x] \tag{10}$$

$$\frac{\partial L}{\partial X}dX + \frac{\partial L}{\partial \dot{X}}d\dot{X} + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} - \varepsilon_x(L[X]).dX = \frac{d}{dt}\theta_L[x] \tag{11}$$

$$\frac{\partial L}{\partial X}dX + \frac{\partial L}{\partial \dot{X}}d\dot{X} + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} - \frac{\partial L}{\partial X}dX + \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}dX - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}dX = \frac{d}{dt}\theta_L[x] \tag{12}$$

$$\frac{\partial L}{\partial \dot{X}}d\dot{X} + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} + \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}dX - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}dX = \frac{d}{dt}\theta_L[x] \tag{13}$$

$$\frac{\partial L}{\partial \dot{X}}d\dot{X} + \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}dX + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}dX = \frac{d}{dt}\theta_L[x] \tag{14}$$

Sol tarafı da tam türev haline getirirsek θ_L i bulabiliriz. (Çarpım türevini kullanarak)

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{X}}dX) + \frac{\partial L}{\partial \ddot{X}}d\ddot{X} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}dX = \frac{d}{dt}\theta_L[x] \tag{15}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{X}}dX) + \frac{d}{dt}(\frac{\partial L}{\partial \ddot{X}}d\dot{X} - dX\frac{d}{dt}(\frac{\partial L}{\partial \ddot{X}})) = \frac{d}{dt}\theta_L[x] \tag{16}$$

$$\frac{d}{dt}\left[(\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}})dX + \frac{\partial L}{\partial \ddot{X}}d\dot{X}\right] = \frac{d}{dt}\theta_L[x] \tag{17}$$

$$\theta_L[x] = \left(\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}}\right)dX + \frac{\partial L}{\partial \ddot{X}}d\dot{X} \tag{18}$$

Buradan yola çıkarak Jacobi-Ostrogradsky momentumları şöyle tanımlanabilir.

$$P^0[X] = \frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} \quad , P^1[X] = \frac{\partial L}{\partial \ddot{X}} \tag{19}$$

Böylece Lagrangian 1-form :

$$\theta_L[X] \equiv P^0[X].dX + P^1[X].d\dot{X} \tag{20}$$

1.2 Reparametrization Invariant Lagrangians

Şimdi 2. dereceden Lagrangian'a sahip sistemin enerjisini bulalım. Bunun için Lagrangian' ın zamana göre türevini alıp Euler-Lagrange denklemini kullanacağız, korunan nicelik de enerji olacak.

$$\frac{dL}{dt} = \frac{\partial L}{\partial X}\dot{X} + \frac{\partial L}{\partial \dot{X}}\ddot{X} + \frac{\partial L}{\partial \ddot{X}}\dddot{X} \tag{21}$$

İlk ifade için Euler lagrange kullanırsak:

$$\frac{dL}{dt} = \dot{X}\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \dot{X}\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} + \frac{\partial L}{\partial \dot{X}}\ddot{X} + \frac{\partial L}{\partial \ddot{X}}\dddot{X} \tag{22}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{X}\frac{\partial L}{\partial \dot{X}}\right) - \dot{X}\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} + \frac{\partial L}{\partial \ddot{X}}\dddot{X} \tag{23}$$

Yine sağdaki ifadeyi çarpım türevi olarak yazabiliriz.

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{X}\frac{\partial L}{\partial \dot{X}}\right) + \frac{d}{dt}\left(-\dot{X}\frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} + \ddot{X}\frac{\partial L}{\partial \ddot{X}}\right) \tag{24}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left((\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}})\dot{X} + \ddot{X}\frac{\partial L}{\partial \ddot{X}}\right) \tag{25}$$

$$0 = \frac{d}{dt}\left((\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}})\dot{X} + \ddot{X}\frac{\partial L}{\partial \ddot{X}} - L[X]\right) \tag{26}$$

$$(\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}})\dot{X} + \ddot{X}\frac{\partial L}{\partial \ddot{X}} - L[X] = sabit \tag{27}$$

Bu sabit enerji olarak tanımlanır.

$$E_L[X] \equiv \dot{X}\cdot\left(\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}}\right) + \ddot{X}\cdot\frac{\partial L}{\partial \ddot{X}} - L[X] \tag{28}$$

Momentumlar cinsinden yazarsak

$$E_L[X] \equiv \dot{X}\cdot P^0[X] + \ddot{X}\cdot P^1[X] - L[X] \tag{29}$$

Tersten gidip Enerjinin zamana göre türevini alalım ve hareket sabiti mi değil mi görelim.

$$\frac{dE_L[X]}{dt} = \ddot{X}\left(\frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}}\right) + \dot{X}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}\right) + \dddot{X}\frac{\partial L}{\partial \ddot{X}} + \ddot{X}\frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} - \frac{\partial L}{\partial X}\dot{X} - \frac{\partial L}{\partial \dot{X}}\ddot{X} - \frac{\partial L}{\partial \ddot{X}}\dddot{X} \tag{30}$$

$\dddot{X}$ lı terimleri götürüp parantezleri açarsak.

$$\frac{dE_L[X]}{dt} = \ddot{X}\frac{\partial L}{\partial \dot{X}} - \ddot{X}\frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} + \dot{X}\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \dot{X}\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} + \ddot{X}\frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} - \frac{\partial L}{\partial X}\dot{X} - \frac{\partial L}{\partial \dot{X}}\ddot{X} \tag{31}$$

$\ddot{X}$ li terimler gider.

$$\frac{dE_L[X]}{dt} = \dot{X}\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} - \dot{X}\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} - \frac{\partial L}{\partial X}\dot{X} \tag{32}$$

$$\frac{dE_L[X]}{dt} = -\dot{X}\cdot\left(\frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}}\right) \tag{33}$$

Euler Lagrange $\frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{X}} = 0$ olduğundan

$$\frac{dE_L[X]}{dt} = -\dot{X}\cdot 0 = 0 \tag{34}$$

Şimdi Lagrangian' ın reparametrizasyon altında invariant olduğunu gösterelim. Yeni parametrizasyon$\tau = \tau(t)$ ve $\lambda \equiv \frac{dt}{d\tau}$, $\nu \equiv \frac{d\lambda}{d\tau} = \frac{d^2t}{d\tau^2}$ olsun. İkinci dereceden Lagrangian aşağıdaki denklemi sağlıyorsa Lagrangian reparametrizasyon invarianttır.

$$\lambda L(X, \dot{X}, \ddot{X}) = L(X, \lambda\dot{X}, \lambda^2\ddot{X} + \nu\dot{X}) \tag{35}$$

$\lambda = 1$ ve $\nu = 0$ eşitlik sağlanmakta. O zaman (35) 'in λ ve ν ye göre türevlerini alıp $(\lambda, \nu) = (1, 0)$ da incelersek infinitesimal invariance şartlarını elde ederiz.

$$L = \frac{\partial L}{\partial X}\frac{\partial X}{\partial \lambda} + \frac{\partial L}{\partial \lambda\dot{X}}\frac{\partial \lambda\dot{X}}{\partial \lambda} + \frac{\partial L}{\partial(\lambda^2\ddot{X} + \nu\dot{X})}\frac{\partial(\lambda^2\ddot{X} + \nu\dot{X})}{\partial \lambda} \tag{36}$$

$$L = \frac{\partial L}{\partial \lambda\dot{X}}\dot{X} + \frac{\partial L}{\partial(\lambda^2\ddot{X} + \nu\dot{X})}2\lambda\ddot{X} \tag{37}$$

$\lambda = 1$ ve $\nu = 0$ kullanılarak ilk şart elde edilir.

$$L = \dot{X} \cdot \frac{\partial L}{\partial \dot{X}} + 2\ddot{X} \cdot \frac{\partial L}{\partial \ddot{X}} \tag{38}$$

ν ye göre türev aldıp aynı noktada incelediğimizde sadece en sondaki terim kalacağından ve sol tarafın ν ye göre türevi 0 olduğundan:

$$0 = \frac{\partial L}{\partial(\lambda^2\ddot{X} + \nu\dot{X})}\frac{\partial(\lambda^2\ddot{X} + \nu\dot{X})}{\partial \nu} = \dot{X} \cdot \frac{\partial L}{\partial \ddot{X}} \tag{39}$$

Böylece Zermelo şartları elde edilir.

$$L = \dot{X} \cdot \frac{\partial L}{\partial \dot{X}} + 2\ddot{X} \cdot \frac{\partial L}{\partial \ddot{X}} \quad , \dot{X} \cdot \frac{\partial L}{\partial \ddot{X}} = 0 \tag{40}$$

Şimdi momentum tanımlarını ve bu şartları kullanarak Lagrangian' ın momentumlar cinsinden ifadesini bulalım. Bunun için momentumların türevlerini ve toplamlarını inceleyelim.

$$P^0[X] + \dot{P}^1[X] = \frac{\partial L}{\partial \dot{X}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} + \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} = \frac{\partial L}{\partial \dot{X}} \tag{41}$$

Denklem (40) 'daki ilk şartın yerine koyarsak

$$L = \dot{X} \cdot (P^0[X] + \dot{P}^1[X]) + 2\ddot{X} \cdot P^1[X] \tag{42}$$

Denklem (40) ’ın ikinci şartından biliyoruz ki $\dot{X} \cdot P^1[X] = \dot{X} \cdot \frac{\partial L}{\partial \ddot{X}} = 0$ bu eşitliği de kullanmak için türevine bakalım:

$$\frac{d}{dt}(P^1[X] \cdot \dot{X}) = \frac{d}{dt}\left(\dot{X} \cdot \frac{\partial L}{\partial \ddot{X}}\right) = \ddot{X} \cdot \frac{\partial L}{\partial \ddot{X}} + \dot{X} \cdot \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} = 0 \tag{43}$$

$$\dot{X} \cdot \frac{d}{dt}\frac{\partial L}{\partial \ddot{X}} = -\ddot{X} \cdot \frac{\partial L}{\partial \ddot{X}} \tag{44}$$

$$\dot{X} \cdot \dot{P}^1[X] = -\ddot{X} \cdot P^1[X] \tag{45}$$

Bu ifadeyi denklem (42) ’yi açıp yerine koyarsak.

$$L = \dot{X} \cdot P^0[X] + \dot{X}\dot{P}^1[X] + 2\ddot{X} \cdot P^1[X] = \dot{X} \cdot P^0[X] - \ddot{X} \cdot P^1[X] + 2\ddot{X} \cdot P^1[X] = \dot{X} \cdot P^0[X] + \ddot{X} \cdot P^1[X] \tag{46}$$

Dolayısıyla

$$L = \dot{X} \cdot P^0[X] + \ddot{X} \cdot P^1[X] \tag{47}$$

Bu Lagrangian (28) ’de yerine konduğunda görüldüğü üzere energy fonksiyonu $E_L[X] = 0$ olmalıdır.

(40) ’deki ikinci ifadenin $\ddot{X}$ e göre türevini aldığımızda $\dot{X}$ e göre denklem sistemi elde ederiz. 0 olmayan çözümlerin varlığı 2. dereceden Lagrangian’ın degenere olmasını ima eder.

$$\dot{X} \cdot \frac{\partial L}{\partial \ddot{X}} = 0 \;\; , \;\; \dot{X} \cdot \frac{\partial^2 L}{\partial \ddot{X}^2} = 0 \tag{48}$$

$$detHess(L) \equiv det\left[\frac{\partial^2 L}{\partial \ddot{X}^2}\right] = 0 \tag{49}$$

2 Hamiltonian Analysis of The Clement Lagrangian

2.1 Clement Lagrangian

$$L^C[X] = -\frac{m}{2}\zeta\dot{X}^2 - \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{2\mu m}X\cdot(\dot{X}\times\ddot{X}) \tag{50}$$

Recall that the Euler-Lagrange equations of the second order Lagrangians are given by the equation (9). Let us compute the required partial derivates and their total time derivatives which will assist us to reach the Euler-Lagrange equations of Clement Lagrangian.

$$\frac{\partial L^C[X]}{\partial X} = A(\dot{X}\times\ddot{X}) \tag{51}$$

$$\frac{d}{dt}\left(\frac{\partial L^C[X]}{\partial \dot{X}}\right) = -m\zeta\ddot{X} + A(\dddot{X}\times X) + A(\ddot{X}\times\dot{X}) \tag{52}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial L^C[X]}{\partial \ddot{X}}\right) = A\Big((\dot{X}\times\ddot{X}) + (X\times\dddot{X})\Big) \tag{53}$$

By putting the equations we have calculated above into equation (9), we have the following Euler-Lagrange equation for Clement Lagrangian.We start with setting $A = \dfrac{\zeta^2}{2\mu m}$.

$$A(\dot{X}\times\ddot{X}) + m\zeta\ddot{X} - A(\dddot{X}\times X) - A(\ddot{X}\times\dot{X}) + A(\dot{X}\times\ddot{X}) + A(X\times\dddot{X}) = 0 \tag{54}$$

$$A(\dot{X}\times\ddot{X}) + m\zeta\ddot{X} + A(X\times\dddot{X}) + A(\dot{X}\times\ddot{X}) + A(\dot{X}\times\ddot{X}) + A(X\times\dddot{X}) = 0 \tag{55}$$

$$m\zeta\ddot{X} + 3A(\dot{X}\times\ddot{X}) + 2A(X\times\dddot{X}) = 0 \tag{56}$$

$$\frac{m\zeta\ddot{X}}{A} + 3(\dot{X}\times\ddot{X}) + 2(X\times\dddot{X}) = 0 \tag{57}$$

$$\frac{2m^2\mu\ddot{X}}{\zeta} + 3(\dot{X}\times\ddot{X}) + 2(X\times\dddot{X}) = 0 \tag{58}$$

If we set the function ζ equal to 1, the third order Euler-Lagrange equation becomes

$$2m^2\mu\ddot{X} + 3(\dot{X}\times\ddot{X}) + 2(X\times\dddot{X}) = 0. \tag{59}$$

In order to get the Lagrangian one-form of Clement Lagrangian, we begin with computing the necessary partial derivates and their total time derivatives.

$$\frac{\partial L^C[X]}{\partial \dot{X}} = -m\zeta\dot{X} + A(\ddot{X}\times X) \tag{60}$$

$$\frac{\partial L^C[X]}{\partial \ddot{X}} = A(X\times\dot{X}) \tag{61}$$

$$\frac{d}{dt}\left(\frac{\partial L^C[X]}{\partial \ddot{X}}\right) = A(X\times\ddot{X}) \tag{62}$$

By putting the equations we have calculated above into equation (18), we have

$$\theta_{L^C}[X] = \Bigg(- m\zeta\dot{X} + A(\ddot{X} \times X) - A(X \times \ddot{X}) \Bigg) \cdot dX + A(X \times \dot{X}) \cdot d\dot{X} \tag{63}$$

$$= -\Bigg(m\zeta\dot{X} + 2A(X \times \ddot{X}) \Bigg) \cdot dX + A(X \times \dot{X}) \cdot d\dot{X} \tag{64}$$

$$= -\Bigg(m\zeta\dot{X} + \frac{\zeta^2}{\mu m}(X \times \ddot{X}) \Bigg) \cdot dX + \frac{\zeta^2}{2\mu m}(X \times \dot{X}) \cdot d\dot{X}. \tag{65}$$

If we substitute the equations (60), (61), and (62) with the guidance of the equation (28), we have

$$E_{L^C}[X] = \dot{X}\Bigg[-m\zeta\dot{X}+\frac{\zeta^2}{2\mu m}(\ddot{X}\times X)-\frac{d}{dt}\Bigg(\frac{\zeta^2}{2\mu m}(X\times\dot{X})\Bigg)\Bigg]+\ddot{X}\Bigg(\frac{\zeta^2}{2\mu m}(X\times\dot{X})\Bigg)-L^C[X]. \tag{66}$$

Now let us compute the term with total time derivative in the equation (66).

Since $\zeta = \zeta(t)$, we apply the product rule.

$$\frac{d}{dt}\Bigg(\frac{\zeta^2}{2\mu m}(X \times \dot{X})\Bigg) = \frac{2\zeta\dot{\zeta}}{2\mu m}(X \times \dot{X}) + \frac{\zeta^2}{2\mu m}\Bigg((\dot{X} \times \dot{X}) + (X \times \ddot{X})\Bigg) \tag{67}$$

$$= \frac{2\zeta\dot{\zeta}}{2\mu m}(X \times \dot{X}) + \frac{\zeta^2}{2\mu m}(X \times \ddot{X}) \tag{68}$$

If we put the result in (68) into (66) we have

$$\dot{X}\Bigg[-m\zeta\dot{X}+\frac{\zeta^2}{2\mu m}(\ddot{X}\times X)-\frac{2\zeta\dot{\zeta}}{2\mu m}(X\times\dot{X})-\frac{\zeta^2}{2\mu m}(X\times\ddot{X})\Bigg]+\ddot{X}\Bigg(\frac{\zeta^2}{2\mu m}(X\times\dot{X})\Bigg)-L^C[X] \tag{69}$$

$$\dot{X}\Bigg[-m\zeta\dot{X}+\frac{\zeta^2}{2\mu m}(\ddot{X}\times X)-\frac{2\zeta\dot{\zeta}}{2\mu m}(X\times\dot{X})+\frac{\zeta^2}{2\mu m}(\ddot{X}\times X)\Bigg]+\frac{\zeta^2}{2\mu m}\dot{X}\cdot(\ddot{X}\times X)-L^C[X] \tag{70}$$

$$\dot{X}\Bigg[- m\zeta\dot{X} + \frac{3\zeta^2}{2\mu m}(\ddot{X} \times X) - \frac{2\zeta\dot{\zeta}}{2\mu m}(X \times \dot{X})\Bigg] - L^C[X] \tag{71}$$

$$- m\zeta\dot{X}^2 + \frac{3\zeta^2}{2\mu m}\dot{X} \cdot (\ddot{X} \times X) - L^C[X]. \tag{72}$$

By replacing $L^C[X]$ by the equation (50), we have

$$E_{L^C}[X] = \frac{-m\zeta\dot{X}^2}{2} + \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{\mu m}\dot{X} \cdot (\ddot{X} \times X). \tag{73}$$

2.2 Dirac-Bergmann Algorithm

Let us find the Jocabi-Ostrogradsky momenta with the guidance of the equations (19), (60), (61),and (62).

$$P^0[X] = -m\zeta\dot{X} + \frac{\zeta^2}{2\mu m}(\ddot{X}\times X) - \frac{d}{dt}\left(\frac{\zeta^2}{2\mu m}(X\times\dot{X})\right) \tag{74}$$

$$= -m\zeta\dot{X} + \frac{\zeta^2}{2\mu m}(\ddot{X}\times X) - \frac{\zeta^2}{2\mu m}(X\times\ddot{X}) \tag{75}$$

$$= -m\zeta\dot{X} + \frac{\zeta^2}{\mu m}(\ddot{X}\times X) \tag{76}$$

$$P^1[X] = \frac{\zeta^2}{2\mu m}(X\times\dot{X}) \tag{77}$$

So, the canonical Hamiltonian function for Clement Lagrangian takes the form

$$H^C = P^0[X]\cdot\dot{X} + P^1[X]\cdot\ddot{X} - L^C \tag{78}$$

$$= -m\zeta\dot{X}^2 + \frac{\zeta^2}{\mu m}\dot{X}\cdot(\ddot{X}\times X) + \frac{\zeta^2}{2\mu m}\ddot{X}\cdot(X\times\dot{X}) + \frac{m}{2}\zeta\dot{X}^2 + \frac{2m\Lambda}{\zeta} - \frac{\zeta^2}{2\mu m}X\cdot(\dot{X}\times\ddot{X}) \tag{79}$$

$$= -\frac{m}{2}\zeta\dot{X}^2 + \frac{2m\Lambda}{\zeta} + P^1[X]\cdot\ddot{X} \tag{80}$$

$$= \frac{m}{2}\zeta\dot{X}^2 + \frac{2m\Lambda}{\zeta} + P^0[X]\cdot\dot{X}. \tag{81}$$

But this is not our total hamiltonian. İn order to construct total hamiltonian we shall apply Dirac constraint analysis. The Ostrogradsky momenta P^1 lead to primary constraint.

$$\mathbf{\Phi} = P^1 - \frac{\zeta^2}{2\mu m}(X\times\dot{X}) \tag{82}$$

The consistency conditions say that.

$$\dot{\mathbf{\Phi}} = \{\mathbf{\Phi}, H^C + \boldsymbol{U}\cdot\mathbf{\Phi}\} \approx 0 \tag{83}$$

In order to calculate above equation, we should see what is poisson bracket for our Darboux coordinates $(X, \dot{X}, P^0, P^1)$. Φ is a vector whereas Hamiltonian is a scalar so we should fix i'th component of Φ and use levi-civita for cross product. For simplicity let $H = H^C + \boldsymbol{U}\cdot\mathbf{\Phi}$

$$\dot{\mathbf{\Phi}}^{\boldsymbol{i}} = \{\mathbf{\Phi}^{\boldsymbol{i}}, H^C + \boldsymbol{U}\cdot\mathbf{\Phi}\} = \sum_j \frac{\partial\mathbf{\Phi}^{\boldsymbol{i}}}{\partial X^j}\frac{\partial H}{\partial P^0_j} - \frac{\partial\mathbf{\Phi}^{\boldsymbol{i}}}{\partial P^0_j}\frac{\partial H}{\partial X^j} + \frac{\partial\mathbf{\Phi}^{\boldsymbol{i}}}{\partial\dot{X}^j}\frac{\partial H}{\partial P^1_j} - \frac{\partial\mathbf{\Phi}^{\boldsymbol{i}}}{\partial P^1_j}\frac{\partial H}{\partial\dot{X}^j} \tag{84}$$

$$\boldsymbol{\Phi^i} = P_i^1 - \frac{\zeta^2}{2\mu m}\varepsilon_{ikl}X_k\dot{X}_l \tag{85}$$

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial X^j} = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikl}\delta_{kj}\dot{X}_l = -\frac{\zeta^2}{2\mu m}\varepsilon_{ijl}\dot{X}_l \tag{86}$$

$$\frac{\partial H}{\partial P_j^0} = \dot{X}_j \tag{87}$$

So the firs term gives:

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial X^j}\frac{\partial H}{\partial P_j^0} = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikl}\delta_{kj}\dot{X}_l = -\frac{\zeta^2}{2\mu m}\varepsilon_{ijl}\dot{X}_j\dot{X}_l = -\frac{\zeta^2}{2\mu m}(\dot{X}\times\dot{X}) = 0 \tag{88}$$

Since $\boldsymbol{\Phi}$ does not depend on P^0 explicitly, the second term is also 0. For the third term:

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial \dot{X}^j} = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikl}\delta_{jl}X_k = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikj}X_k \tag{89}$$

$$\frac{\partial H}{\partial P_j^1} = U_j \tag{90}$$

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial \dot{X}^j}\frac{\partial H}{\partial P_j^1} = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikl}\delta_{jl}X_k = -\frac{\zeta^2}{2\mu m}\varepsilon_{ikj}X_kU_j = -\frac{\zeta^2}{2\mu m}(X\times\boldsymbol{U}) = \frac{\zeta^2}{2\mu m}(\boldsymbol{U}\times X) \tag{91}$$

And the last term:

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial P_j^1} = 1 \tag{92}$$

$$\frac{\partial H}{\partial \dot{X}^j} = m\zeta\dot{X}^j + P_j^0 - \frac{\zeta^2}{2\mu m}\frac{\partial}{\partial \dot{X}^j}(\varepsilon_{ikl}U_iX_k\dot{X}_l) = m\zeta\dot{X}^j + P_j^0 - \frac{\zeta^2}{2\mu m}\frac{\partial}{\partial \dot{X}^j}(\varepsilon_{lik}U_iX_k\dot{X}_l) \tag{93}$$

When we sum it for all j instead of fixed.

$$= m\zeta\dot{X} + P^0 - \frac{\zeta^2}{2\mu m}(\boldsymbol{U}\times X) \tag{94}$$

So the last term becomes

$$-\frac{\partial \boldsymbol{\Phi^i}}{\partial P_j^1}\frac{\partial H}{\partial \dot{X}^j} = -m\zeta\dot{X} - P^0 + \frac{\zeta^2}{2\mu m}(\boldsymbol{U}\times X) \tag{95}$$

When we sum for all components our poisson bracket becomes:

$$\{\boldsymbol{\Phi}, H^C + \boldsymbol{U}\cdot\boldsymbol{\Phi}\} = \frac{\zeta^2}{2\mu m}(U\times X) - m\zeta\dot{X} - P^0 + \frac{\zeta^2}{2\mu m}(\boldsymbol{U}\times X) = -m\zeta\dot{X} - P^0 + \frac{\zeta^2}{\mu m}(\boldsymbol{U}\times X) \approx 0 \tag{96}$$

Due to the degeneracy of the cross product, only two components of the Lagrange multipliers $\boldsymbol{U}$ can be solved from these equations and a secondary constraint

$$\Phi = m\zeta(\boldsymbol{X}\cdot\dot{\boldsymbol{X}}) + \boldsymbol{X}\cdot\boldsymbol{P^0} \tag{97}$$

arises.We add the secondary constraint to the Hamiltonian function and define the total Hamiltonian as

$$H_T^C = H^C + \boldsymbol{U} \cdot \boldsymbol{\Phi} + U \cdot \Phi \tag{98}$$

Now, let us find consistency conditions by using the total Hamiltonian H_T^C.

$$\dot{\Phi} = \{\Phi, H_T^C\} = \frac{\partial \Phi}{\partial X} \cdot \frac{\partial H_T^C}{\partial P^0} - \frac{\partial \Phi}{\partial P^0} \cdot \frac{\partial H_T^C}{\partial X} + \frac{\partial \Phi}{\partial \dot{X}} \cdot \frac{\partial H_T^C}{\partial P^1} - \frac{\partial \Phi}{\partial P^1} \cdot \frac{\partial H_T^C}{\partial \dot{X}} \tag{99}$$

Calculating term by term, we have:

$$\frac{\partial \Phi}{\partial X} \cdot \frac{\partial H_T^C}{\partial P^0} = (m\zeta \dot{X} + P^0) \cdot (\dot{X} + U \cdot X) \tag{100}$$

$$\frac{\partial \Phi}{\partial P^0} \cdot \frac{\partial H_T^C}{\partial X} = X \cdot [(m\zeta \dot{X} + P^0) \cdot U - \frac{\zeta^2}{2\mu m}(\dot{X} \times \boldsymbol{U})] \tag{101}$$

$$\frac{\partial \Phi}{\partial \dot{X}} \cdot \frac{\partial H_T^C}{\partial P^1} = m\zeta X \cdot \boldsymbol{U}, \qquad \frac{\partial \Phi}{\partial P^1} \cdot \frac{\partial H_T^C}{\partial \dot{X}} = 0 \tag{102}$$

After using the equation (77) and arranging the terms we have:

$$\dot{\Phi} = \dot{\boldsymbol{X}} \cdot \boldsymbol{B} + \boldsymbol{U} \cdot \boldsymbol{A} \tag{103}$$

Here, we used the abbreviations

$$\boldsymbol{A} = m\zeta \boldsymbol{X} + \boldsymbol{P^1} \quad \textit{and} \quad \boldsymbol{B} = m\zeta \dot{\boldsymbol{X}} + \boldsymbol{P^0} \tag{104}$$

Now let us compute $\boldsymbol{\Phi}$ by using H_T^C

$$\dot{\boldsymbol{\Phi}} = \{\boldsymbol{\Phi}, H_T^C\} = \{\boldsymbol{\Phi}, H^C + \boldsymbol{U} \cdot \boldsymbol{\Phi} + U \cdot \Phi\} \tag{105}$$

Since it is a linear operator

$$\{\boldsymbol{\Phi}, H_T^C\} = \{\boldsymbol{\Phi}, H^C + \boldsymbol{U} \cdot \boldsymbol{\Phi}\} + \{\boldsymbol{\Phi}, U \cdot \Phi\} \tag{106}$$

Since we already calculated the first term, let us calculate the second one

$$\{\boldsymbol{\Phi^i}, U \cdot \Phi\} = \sum_j \frac{\partial \boldsymbol{\Phi^i}}{\partial X^j} \frac{\partial U \cdot \Phi}{\partial P_j^0} - \frac{\partial \boldsymbol{\Phi^i}}{\partial P_j^0} \frac{\partial U \cdot \Phi}{\partial X^j} + \frac{\partial \boldsymbol{\Phi^i}}{\partial \dot{X}^j} \frac{\partial U \cdot \Phi}{\partial P_j^1} - \frac{\partial \boldsymbol{\Phi^i}}{\partial P_j^1} \frac{\partial U \cdot \Phi}{\partial \dot{X}^j} \tag{107}$$

$$\boldsymbol{\Phi^i} = P_i^1 - \frac{\zeta^2}{2\mu m} \varepsilon_{ikl} X_k \dot{X}_l \tag{108}$$

$$\frac{\partial \boldsymbol{\Phi^i}}{\partial X^j} = -\frac{\zeta^2}{2\mu m} \varepsilon_{ikl} \delta_{kj} \dot{X}_l = -\frac{\zeta^2}{2\mu m} \varepsilon_{ijl} \dot{X}_l, \qquad \frac{\partial U \cdot \Phi}{\partial P_j^0} = X_j \tag{109}$$

So, we can write first term as

$$-\frac{\zeta^2}{2\mu m}(X \times \dot{X}) \tag{110}$$

Since $\mathbf{\Phi}$ does not depend on P^0 and Φ does not depend on P^1 second and third terms are zero. So, last term can be written as

$$-\frac{\partial \mathbf{\Phi}^{\boldsymbol{i}}}{\partial P_j^1}\frac{\partial U\cdot\Phi}{\partial \dot{X}^j} = -(1)\cdot m\zeta\boldsymbol{X} \tag{111}$$

Finally we can arrange terms and write

$$\dot{\mathbf{\Phi}} = \{\mathbf{\Phi}, H_T^C\} = -m\zeta\dot{\boldsymbol{X}} - \boldsymbol{P^0} + \frac{\zeta^2}{\mu m}\boldsymbol{U}\times\boldsymbol{X} - U\boldsymbol{A} \tag{112}$$

Under the assumption of $X^2 = 0$, we solve the Lagrange multipliers U and $\boldsymbol{U}$ as follows First to calculate U let us use $\dot{\mathbf{\Phi}}\cdot\boldsymbol{X}$, which gives

$$m\zeta\boldsymbol{X}\cdot\dot{\boldsymbol{X}} + \boldsymbol{P^0}\boldsymbol{\cdot}\boldsymbol{X} = -mX^2\zeta U \tag{113}$$

$$U = \frac{-1}{mX^2\zeta}\boldsymbol{X}\boldsymbol{\cdot}\boldsymbol{B} = \frac{-1}{mX^2\zeta}\Phi \tag{114}$$

To calculate $\boldsymbol{U}$ we will use $\dot{\mathbf{\Phi}}\times\boldsymbol{A}$ and solve equation system by using $\dot{\mathbf{\Phi}}\times\boldsymbol{A}$ and $\dot{\Phi}$X

$$\dot{\mathbf{\Phi}}\times\boldsymbol{A} - \frac{\zeta^2}{\mu m}(\Phi\boldsymbol{X}) = \boldsymbol{A}\times\boldsymbol{B} + \frac{\zeta^2}{m\mu}\boldsymbol{(U}\times\boldsymbol{X)}\times\boldsymbol{A} - U\boldsymbol{A}\times\boldsymbol{A} - \frac{\zeta^2}{\mu m}((\boldsymbol{X}\boldsymbol{\cdot}\boldsymbol{B}X + (\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{A})X) \tag{115}$$

$$\frac{\zeta^3X^2}{m}\boldsymbol{U} = -(\boldsymbol{B}\times\boldsymbol{A} + \frac{\zeta^2}{\mu m}(\dot{\boldsymbol{X}}\boldsymbol{\cdot}\boldsymbol{B})X) \tag{116}$$

$$\boldsymbol{B}\times\boldsymbol{A} = m\zeta\boldsymbol{B}\times\boldsymbol{X} + \frac{\zeta^2}{2\mu m}[(\boldsymbol{B}\boldsymbol{\cdot}\dot{\boldsymbol{X}})X - (\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{X})\dot{\boldsymbol{X}}] \tag{117}$$

Since $\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{X} = 0$, we have

$$\boldsymbol{U} = \frac{-3}{2m\zeta X^2}(\boldsymbol{B}\boldsymbol{\cdot}\dot{\boldsymbol{X}})\boldsymbol{X} - \frac{\mu m}{\zeta^2X^2}(\boldsymbol{B}\times\boldsymbol{X}) \tag{118}$$

After substitution of the constraints $\mathbf{\Phi}$ and Φ, and the Lagrange multipliers $\boldsymbol{U}$ and U into implicit form of H_T^C we write the total Hamiltonian as

$$H_T^C = \frac{1}{2}\dot{\boldsymbol{X}}\boldsymbol{\cdot}\boldsymbol{P^0} - \frac{3}{2m\zeta X^2}(\boldsymbol{X}\boldsymbol{\cdot}\boldsymbol{P^1})(\boldsymbol{B}\boldsymbol{\cdot}\dot{\boldsymbol{X}}) - \frac{\mu m}{\zeta^2X^2}\boldsymbol{P^1}\cdot(\boldsymbol{B}\times\boldsymbol{X}) + \frac{1}{2X^2}(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{X})(\boldsymbol{X}\boldsymbol{\cdot}\dot{\boldsymbol{X}}) - \frac{1}{m\zeta X^2}(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{X})^2 \tag{119}$$

Using H_T^C and canonical Poisson bracket 107 we can compute the Hamilton's equations: (Using the equality for B) (104) Only partial X remains

$$\{X, H_T^C\} = \dot{X} \approx \frac{\partial X}{\partial X}\frac{\partial H_T^C}{\partial P^0} = \frac{\partial H_T^C}{\partial P^0} \tag{120}$$

the term $\frac{3}{2m\zeta X^2}(X \cdot P^1)\dot{X}$ gives 0 since $X \cdot P^1 = 0$ and also the term $\frac{-1}{m\zeta X^2}(B \cdot X)^2$ also vanishes as $B \cdot X = 0$ so we are left with:

$$\dot{X} \approx \frac{1}{2}\dot{X} + \frac{1}{2X^2}X(X \cdot \dot{X}) + \frac{\mu m}{\zeta^2 X^2}(P^1 \times X) \tag{121}$$

$$\{\dot{X}, H_T^C\} = \ddot{X} \approx \frac{\partial \dot{X}}{\partial \dot{X}}\frac{\partial H_T^C}{\partial P^1} \tag{122}$$

So we have:

$$\ddot{X} \approx -\frac{3}{2m\zeta X^2}(B \cdot \dot{X})X + \frac{\mu m}{\zeta^2 X^2}(X \times B) \tag{123}$$

$$\{P^1, H_T^C\} = \dot{P}^1 \approx -\frac{\partial P^1}{\partial P^1}\frac{\partial H}{\partial \dot{X}} = -\frac{\partial H}{\partial \dot{X}} - \frac{\partial H}{\partial B}\frac{\partial B}{\partial \dot{X}} = -\frac{\partial H}{\partial \dot{X}} - \frac{\partial H}{\partial B}m\zeta \tag{124}$$

$$\dot{P}^1 \approx -\frac{1}{2}P^0 - \frac{\mu m^2}{\zeta X^2}(P^1 \times X) - \frac{m\zeta}{2X^2}X(X \cdot \dot{X}) \tag{125}$$

for P^0:

$$\{P^0, H_T^C\} = \dot{P}^0 \approx -\frac{\partial P^0}{\partial P^0}\frac{\partial H}{\partial X} = -\frac{\partial H}{\partial X} \tag{126}$$

$$\dot{P}^0 \approx \frac{3}{2m\zeta X^2}P^1(B \cdot \dot{X}) - \frac{1}{2X^2}(\dot{X} \cdot X)B + \frac{\mu m}{\zeta^2 X^2}P^1 \times B - \frac{2}{X^4}\frac{\mu m}{\zeta^2}(P^1.B \times X)X \tag{127}$$

It is easy to see that $\ddot{X} \approx \boldsymbol{U}$ From the equation of P^0 above, the article derives

$$\dot{P}^0 = \frac{\zeta^2}{2m\mu}(\dot{X} \times \ddot{X}) \tag{128}$$

(74) using the momenta definition for P^0, we take the derivative and use the above equation

$$\dot{P}^0 = -m\zeta\ddot{X} + \frac{\zeta^2}{\mu m}(\ddot{X} \times \dot{X}) + \frac{\zeta^2}{\mu m}(\dddot{X} \times X) \tag{129}$$

$$-m\zeta\ddot{X} + \frac{\zeta^2}{\mu m}(\ddot{X} \times \dot{X}) + \frac{\zeta^2}{\mu m}(\dddot{X} \times X) = \frac{\zeta^2}{2m\mu}(\dot{X} \times \ddot{X}) \tag{130}$$

$$-m\zeta\ddot{X} + \frac{3\zeta^2}{2\mu m}(\ddot{X} \times \dot{X}) + \frac{\zeta^2}{\mu m}(\dddot{X} \times X) = 0 \tag{131}$$

multiplying both sides by $\frac{2m\mu}{\zeta}$ and rearranging give (59) Euler Lagrange equation for the Clement Lagrangian

3 Hamiltonian Analysis of The Sarıoğlu-Tekin Lagrangian

3.1 Sarıoğlu-Tekin Lagrangian

$$L^{ST}[X,Y] = \frac{1}{2}\left[a(\dot{X}^2 + \dot{Y}^2) + \frac{2}{\mu}\dot{Y}\cdot\ddot{X} - m^2(Y^2 + X^2)\right] \tag{132}$$

Let us compute the required partial derivates and their total time derivatives which we shall use to reach the Euler-Lagrange equations of Sarıoğlu-Tekin Lagrangian.

$$\frac{\partial L^{ST}[X,Y]}{\partial X} = -m^2 X \tag{133}$$

$$\frac{\partial L^{ST}[X,Y]}{\partial Y} = -m^2 Y \tag{134}$$

$$\frac{d}{dt}\left(\frac{\partial L^{ST}[X,Y]}{\partial \dot{X}}\right) = a\ddot{X} \tag{135}$$

$$\frac{d}{dt}\left(\frac{\partial L^{ST}[X,Y]}{\partial \dot{Y}}\right) = a\ddot{Y} + \frac{\dddot{X}}{\mu} \tag{136}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial L^{ST}[X,Y]}{\partial \ddot{X}}\right) = \frac{\dddot{Y}}{\mu} \tag{137}$$

Since $L^{ST}[X]$ does not have any term that include $\ddot{Y}$, it does not have a partial derivative with respect to $\ddot{Y}$. We found that the Euler-Lagrange equations of second order Lagrangians are given by the equation (9). By putting the equations we have calculated above into equation (9), we have the following Euler-Lagrange equations for Sarıoğlu-Tekin Lagrangian.

$$-m^2 X - a\ddot{X} + \frac{\dddot{Y}}{\mu} = 0 \tag{138}$$

$$m^2 Y + a\ddot{Y} + \frac{\dddot{X}}{\mu} = 0 \tag{139}$$

In order to get the Lagrangian one-form of Sarıoğlu-Tekin Lagrangian, we begin with computing the necessary partial derivates and their total time derivatives.

$$\frac{\partial L^{ST}[X,Y]}{\partial \dot{X}} = a\dot{X} \tag{140}$$

$$\frac{\partial L^{ST}[X,Y]}{\partial \dot{Y}} = a\dot{Y} + \frac{\ddot{X}}{\mu} \tag{141}$$

$$\frac{\partial L^{ST}[X,Y]}{\partial \ddot{X}} = \frac{\dot{Y}}{\mu} \tag{142}$$

$$\frac{d}{dt}\left(\frac{\partial L^{ST}[X,Y]}{\partial \ddot{X}}\right) = \frac{\ddot{Y}}{\mu} \tag{143}$$

By putting the equations we have calculated above into equation (18), we have

$$\theta_{L^{ST}}[X,Y] = \left(a\dot{X} - \frac{\ddot{Y}}{\mu}\right) \cdot dX + \left(a\dot{Y} + \frac{\ddot{X}}{\mu}\right) \cdot dY + \frac{\dot{Y}}{\mu} \cdot d\dot{X}. \tag{144}$$

Recall the energy of the system of second order Lagrangians is given by the equation (28). By putting the equations (140), (141), (142), and (143) into the equation (28), we have the following energy for Sarıoğlu-Tekin Lagrangian system.

$$E_{L^{ST}}[X,Y] = \dot{X} \cdot \left(a\dot{X} - \frac{\ddot{Y}}{\mu}\right) - \frac{\dot{Y} \cdot \ddot{X}}{\mu} + \dot{Y} \cdot \left(a\dot{Y} + \frac{\ddot{X}}{\mu}\right) - L^{ST}[X,Y] \tag{145}$$

$$= a(\dot{X}^2 + \dot{Y}^2) - \frac{\dot{X} \cdot \ddot{Y}}{\mu} - L^{ST}[X,Y] \tag{146}$$

By substituting the equation (132) for $L^{ST}[X,Y]$ and rearranging, the expression becomes

$$E_{L^{ST}}[X,Y] = a(\dot{X}^2 + \dot{Y}^2) - \frac{\dot{X} \cdot \ddot{Y}}{\mu} - \frac{a}{2}(\dot{X}^2 + \dot{Y}^2) - \frac{\ddot{X} \cdot \dot{Y}}{\mu} + \frac{m^2}{2}(Y^2 + X^2) \tag{147}$$

$$= \frac{a}{2}(\dot{X}^2 + \dot{Y}^2) - \frac{1}{\mu}(\dot{X} \cdot \ddot{Y} + \ddot{X} \cdot \dot{Y}) + \frac{m^2}{2}(Y^2 + X^2) \tag{148}$$

What we have found here is quiet different from the article. Hence, we have to spend more time on it.

3.2 Dirac-Bergmann Algorithm

Let us find the Jocabi-Ostrogradsky momenta with the guidance of the equations (19), (141), (141), (142), and (143).

$$P_X^0 = a\dot{X} - \frac{\ddot{Y}}{\mu} \tag{149}$$

$$P_Y^0 = a\dot{Y} + \frac{\ddot{X}}{\mu} \tag{150}$$

$$P_X^1 = \frac{\dot{Y}}{\mu} \tag{151}$$

$$P_Y^1 = 0 \tag{152}$$

And the corresponding canonical Hamiltonian function for Sarıoğlu-Tekin Lagrangian is

$$H^{ST} = P_X^0 \cdot \dot{X} + P_Y^0 \cdot \dot{Y} + P_X^1 \cdot \ddot{X} + P_Y^1 \cdot \ddot{Y} - L^{ST} \tag{153}$$

$$= P_X^0 \cdot \dot{X} + P_Y^0 \cdot \dot{Y} + \frac{\ddot{X} \cdot \dot{Y}}{\mu} - \frac{a}{2}(\dot{X}^2 + \dot{Y}^2) - \frac{\ddot{X} \cdot \dot{Y}}{\mu} + \frac{m^2}{2}(Y^2 + X^2) \tag{154}$$

$$= P_X^0 \cdot \dot{X} + P_Y^0 \cdot \dot{Y} - \frac{a}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m^2}{2}(Y^2 + X^2) \tag{155}$$

The expressions of momenta in the equations (149) and (150) enable us to solve 2 second derivatives.

$$\ddot{Y} = \mu(a\dot{X} - P_X^0), \;\; \ddot{X} = \mu(P_Y^0 - a\dot{Y}) \tag{156}$$

Furthermore, the equations (151) and (152) imply the set of primary constraints.

$$\boldsymbol{\Phi} = P_X^1 - \frac{1}{\mu}\dot{Y} \approx 0, \;\; \boldsymbol{\Psi} = P_Y^1 \approx 0 \tag{157}$$

Let us then introduce total Hamiltonian H_T^{ST} as the sum of H^{ST} the Eq.(155) and the primary constraint in the Eq.(157).

$$H_T^{ST} = H^{ST} + \boldsymbol{U} \cdot \boldsymbol{\Phi} + \boldsymbol{V} \cdot \boldsymbol{\Psi} \tag{158}$$

Where $\boldsymbol{U}$ and $\boldsymbol{V}$ are Lagrange multipliers. The consistency conditions occasion determination of the Lagrange multipliers

$$\boldsymbol{V} = \mu(a\dot{X} - P_X^0), \;\; \boldsymbol{U} = \mu(P_Y^0 - a\dot{Y}) \tag{159}$$

without implying any new constraint.

$$\boldsymbol{U} \cdot \boldsymbol{\Phi} = \mu\Big(P_Y^0 - a\dot{Y}\Big) \cdot \Big(P_X^1 - \frac{\dot{Y}}{\mu}\Big) \tag{160}$$

$$= \mu P_Y^0 \cdot P_X^1 - P_Y^0 \cdot \dot{Y} - a\mu\dot{Y}P_X^1 + a\dot{Y}^2 \tag{161}$$

$$\boldsymbol{V} \cdot \boldsymbol{\Psi} = \mu(a\dot{X} - P_X^0) \cdot P_Y^1 \tag{162}$$

$$= \mu a\dot{X} \cdot P_Y^1 - \mu P_X^0 \cdot P_Y^1 \tag{163}$$

By substituting $\boldsymbol{U}\cdot\boldsymbol{\Phi}$ in Eq.(161), $\boldsymbol{V}\cdot\boldsymbol{\Psi}$ in Eq.(163), and H^{ST} in Eq.(155) into H_T^{ST} in Eq.(158), we obtain the total Hamiltonian function of Dirac

$$
\begin{aligned}
H_T^{ST} &= P_X^0\cdot\dot{X} + P_Y^0\cdot\dot{Y} - \frac{a}{2}(\dot{X}^2+\dot{Y}^2) + \frac{m^2}{2}(Y^2+X^2) && (164)\\
&+ \mu P_Y^0\cdot P_X^1 - P_Y^0\cdot\dot{Y} - a\mu\dot{Y}P_X^1 + a\dot{Y}^2 && (165)\\
&+ \mu a\dot{X}\cdot P_Y^1 - \mu P_X^0\cdot P_Y^1 && (166)\\
&= \mu(P_Y^0\cdot P_X^1 - P_X^0\cdot P_Y^1) + a\mu(\dot{X}\cdot P_Y^1 - \dot{Y}P_X^1) + P_X^0\cdot\dot{X} - \frac{a}{2}(\dot{X}^2-\dot{Y}^2) + \frac{m^2}{2}(Y^2+X^2). && (167)
\end{aligned}
$$

In order to obtain Hamilton's equations, let us compute the required partial derivatives in advance.

$$
\begin{aligned}
\frac{\partial(\boldsymbol{U}\cdot\boldsymbol{\Phi})}{\partial P_Y^0} &= \mu P_X^1 - \dot{Y} && (168)\\
\frac{\partial(\boldsymbol{V}\cdot\boldsymbol{\Psi})}{\partial P_X^0} &= -\mu P_Y^1 && (169)\\
\frac{\partial H^{ST}}{\partial P_Y^0} &= \dot{Y} && (170)\\
\frac{\partial H^{ST}}{\partial P_X^0} &= \dot{X} && (171)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H_T^{ST}}{\partial P_Y^0} &= \frac{\partial H^{ST}}{\partial P_Y^0} + \frac{\partial(\boldsymbol{U}\cdot\boldsymbol{\Phi})}{\partial P_Y^0} + \frac{\partial(\boldsymbol{V}\cdot\boldsymbol{\Psi})}{\partial P_Y^0} && (172)\\
\frac{\partial H_T^{ST}}{\partial P_X^0} &= \frac{\partial H^{ST}}{\partial P_X^0} + \frac{\partial(\boldsymbol{U}\cdot\boldsymbol{\Phi})}{\partial P_X^0} + \frac{\partial(\boldsymbol{V}\cdot\boldsymbol{\Psi})}{\partial P_X^0} && (173)
\end{aligned}
$$

Now, we can obtain the partial derivatives of H_T^{ST} with respect to P_Y^0 and P_X^0 by substituting the partial derivatives we computed above into the Eq.(172) and Eq.(173). So we have,

$$
\begin{aligned}
\frac{\partial H_T^{ST}}{\partial P_Y^0} &= \mu P_X^1 && (174)\\
\frac{\partial H_T^{ST}}{\partial P_X^0} &= \dot{X} - \mu P_Y^1 && (175)
\end{aligned}
$$

For the base variables $(X, Y, \dot{X}, \dot{Y})$, the equations of motion governed by the total Hamiltonian function H_T^{ST} are

$$
\begin{aligned}
\ddot{Y} &= \mu a\dot{X} - \mu P_X^0 && (176)\\
\ddot{X} &= \mu P_Y^0 - a\mu\dot{Y} && (177)\\
\dot{Y} &= \{Y, H_T^{ST}\} && (178)\\
\dot{X} &= \{X, H_T^{ST}\} && (179)
\end{aligned}
$$

where we used the equations in (156) to obtain Eq.(176) and Eq.(177). Now, let us take Eq.(178) and Eq.(179) into consideration.

$$\begin{aligned}\dot{Y} &= \{Y, H_T^{ST}\} \quad (180)\\ &= \frac{\partial Y}{\partial Y}\cdot\frac{\partial H_T^{ST}}{\partial P_Y^0} - \frac{\partial Y}{\partial P_Y^0}\cdot\frac{\partial H_T^{ST}}{\partial Y} + \frac{\partial Y}{\partial \dot{Y}}\cdot\frac{\partial H_T^{ST}}{\partial P_Y^1} - \frac{\partial Y}{\partial P_Y^1}\cdot\frac{\partial H_T^{ST}}{\partial \dot{Y}} \quad (181)\end{aligned}$$

It can be obviously seen that only the first term remains in Eq.(181) (This argument also holds for Eq.(184)). So we have

$$\dot{Y} = \mu P_X^1 \quad (182)$$

by substituting Eq.(174).

$$\begin{aligned}\dot{X} &= \{X, H_T^{ST}\} \quad (183)\\ &= \frac{\partial X}{\partial X}\cdot\frac{\partial H_T^{ST}}{\partial P_X^0} - \frac{\partial X}{\partial P_X^0}\cdot\frac{\partial H_T^{ST}}{\partial X} + \frac{\partial X}{\partial \dot{X}}\cdot\frac{\partial H_T^{ST}}{\partial P_X^1} - \frac{\partial X}{\partial P_X^1}\cdot\frac{\partial H_T^{ST}}{\partial \dot{X}} \quad (184)\end{aligned}$$

By substituting Eq.(175), we have

$$\dot{X} = \dot{X} - \mu P_Y^1. \quad (185)$$

For momenta $(P_X^0, P_Y^0, P_X^1, P_Y^1)$, the equations of motion are the following.

$$\begin{aligned}\dot{P_X^0} &= \{P_X^0, H_T^{ST}\} \quad (186)\\ \dot{P_Y^0} &= \{P_Y^0, H_T^{ST}\} \quad (187)\\ \dot{P_X^1} &= \{P_X^1, H_T^{ST}\} \quad (188)\\ \dot{P_Y^1} &= \{P_Y^1, H_T^{ST}\} \quad (189)\end{aligned}$$

Before computing the Poisson brackets above one by one, let us first determine the partial derivatives of H_T^{ST}, which will assist us during calculation. We have the equations

$$\begin{aligned}\frac{\partial H_T^{ST}}{\partial X} &= m^2 X \quad (190)\\ \frac{\partial H_T^{ST}}{\partial Y} &= m^2 Y \quad (191)\\ \frac{\partial H_T^{ST}}{\partial \dot{X}} &= -a\dot{X} + P_X^0 + a\mu P_Y^1 \quad (192)\\ \frac{\partial H_T^{ST}}{\partial \dot{Y}} &= a\dot{Y} - a\mu P_X^1 \quad (193)\end{aligned}$$

by using the Eq.(167). So we have the equations of motion

$$
\begin{aligned}
\dot{P_X^0} &= \{P_X^0, H_T^{ST}\} && (194)\\
&= \frac{\partial P_X^0}{\partial X}\cdot\frac{\partial H_T^{ST}}{\partial P_X^0} - \frac{\partial P_X^0}{\partial P_X^0}\cdot\frac{\partial H_T^{ST}}{\partial X} && (195)\\
&= -m^2 X && (196)\\
\dot{P_Y^0} &= \{P_Y^0, H_T^{ST}\} && (197)\\
&= \frac{\partial P_Y^0}{\partial Y}\cdot\frac{\partial H_T^{ST}}{\partial P_Y^0} - \frac{\partial P_Y^0}{\partial P_Y^0}\cdot\frac{\partial H_T^{ST}}{\partial Y} && (198)\\
&= -m^2 Y && (199)\\
\dot{P_X^1} &= \{P_X^1, H_T^{ST}\} && (200)\\
&= \frac{\partial P_X^1}{\partial \dot{X}}\cdot\frac{\partial H_T^{ST}}{\partial P_X^1} - \frac{\partial P_X^1}{\partial P_X^1}\cdot\frac{\partial H_T^{ST}}{\partial \dot{X}} && (201)\\
&= a\dot{X} - P_X^0 - a\mu P_Y^1 && (202)\\
\dot{P_Y^1} &= \{P_Y^1, H_T^{ST}\} && (203)\\
&= \frac{\partial P_Y^1}{\partial \dot{Y}}\cdot\frac{\partial H_T^{ST}}{\partial P_Y^1} - \frac{\partial P_Y^1}{\partial P_Y^1}\cdot\frac{\partial H_T^{ST}}{\partial \dot{Y}} && (204)\\
&= -a\dot{Y} + a\mu P_X^1 && (205)
\end{aligned}
$$

where we used the equations (190), (191), (192), and (193).

3.3 Dirac-Poisson Bracket