PHYS301: Final exam

October 30, 2022

Problem 1. Consider a particle of mass m moving along the cycloid curve under the influence of gravity. The cycloid curve can be parametrized via parameter u as follows

$$x = a(u - \sin u), \quad y = -a(1 - \cos u) \ ,$$

where a is a constant.

- Write the Lagrangian in terms of the generalized coordinate u.
- Introduce a new coordinate $s = \cos(u/2)$ and obtain the equation of motion in terms of the coordinate s and solve the equation of motion.
- Calculate the energy of the system in terms of the coordinate s.

Problem 2. Obtain the Lagrangian from the following Hamiltonian

$$H(\mathbf{p}, \mathbf{r}) = \frac{(\mathbf{p} \cdot \mathbf{p})}{2m} - (\mathbf{p} \cdot \mathbf{a})$$

where $\mathbf{a}$ is a constant vector and $(\mathbf{a} \cdot \mathbf{b})$ is a scalar product.

Problem 3a. Find the ratio of times (in terms of ratio of masses) in the same path for particles having different masses, but the same potential energy.

Problem 3b. A particle of mass m, moving with velocity v_1 leaves a half-space in which its potential energy is a constant U_1, and enters the other half-space, in which its potential energy is another constant U_2. In the first half-space the velocity of the particle makes an angle θ_1 with the normal to the plane separating the two half-spaces, in the second it makes an angle θ_2 with the normal. Determine the ratio of $\frac{\sin\theta_1}{\sin\theta_2}$.

Problem 4a.

- Find under which condition the following transformation is canonical

$$\begin{aligned} Q &= a_{11} q + a_{12} p \\ P &= a_{21} q + a_{22} p \end{aligned}$$

- Write the generating function $F(q, P)$ for this transformation.

Problem 4b. For a free particle $H = \frac{p^2}{2m}$ show that if

$$Q = \frac{1}{2} m q^2 - qpt, \quad P = p$$

then $\{Q, P\}$ is an integral of motion (constant of motion).

Problem 5. The Lagrangian of the Kepler's problem has the following form

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}}, \quad \alpha > 0$$

- Obtain the Hamiltonian of the Kepler's problem in the elliptic coordinates ξ, η and ϕ

$$x = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos\phi, \quad y = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin\phi, \quad z = \xi\eta + 1$$

- Solve the equations of motion of the Kepler's problem in the elliptic coordinates ξ, η and ϕ via the Hamilton-Jacobi equation.