PHYS302: Midterm I

March 20, 2023

Problem 1a. For the singular Lagrangian

$$L = \frac{1}{2}\dot{x}_1^2 - \dot{x}_2 x_3$$

find the primary constraints of the system.

Problem 1b. Find the frequency of small oscillations of the following system

$$L = \dot{x}^2 - x^2 e^x.$$

Problem 2. Show that a damped harmonic oscillator

$$\ddot{x} + \alpha\dot{x} + \omega^2 x = 0$$

can be described by a Lagrangian.

Problem 3. Show that the eigenvectors of the symmetric matrix have the property that eigenvectors with different eigenvalues are orthogonal.

Problem 4. The Lagrangian of the double pendulum has the following form

$$L = \frac{1}{2}ml^2(2\dot{\phi}_1^2 + 2\dot{\phi}_1(\dot{\phi}_1 + \dot{\phi}_2)\cos\phi_2 + (\dot{\phi}_1 + \dot{\phi}_2)^2) + mgl(2\cos\phi_1 + \cos(\phi_1 + \phi_2))$$

For the small angle approximation find the normal mode frequencies.

Problem 5.

The Lagrangian of the time-dependent harmonic oscillator is given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2.$$

By using the Noether theorem, show that there is an integral of motion

$$I(t) = \frac{1}{2}\left((\rho p - \dot{\rho}x)^2 + \frac{x^2}{\rho^2}\right),$$

where $\rho(t)$ is an auxiliary variable satisfying

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}$$

Hint. Note that under the following transformations

$$x' = x + \eta(x,t)\epsilon$$

$$t' = t + \xi(x,t)\epsilon$$

there is an expression (where $f(x,t)$ is a total derivative of some function)

$$\dot{\xi}L + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{\xi}\dot{x})\frac{\partial L}{\partial \dot{x}} + \xi\frac{\partial L}{\partial t} = \dot{f}$$

and the integral of motion has the following form

$$I = (\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f\ .$$