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Problem 1. Let A be a 2×2 matrix with $\det(A) = 1$. Using the Cayley-Hamilton theorem, show that

$$A^{-1} + A = \mathrm{Tr}(A)\, I,$$

where $\mathrm{Tr}(A)$ denotes the trace of A, and I is the 2×2 identity matrix.

Problem 2. Consider the system with the Hamiltonian

$$H = \dot{x}^2 - \left(e^x - 1\right)^2.$$

Find the period of the motion of this system.

Problem 3. Consider a rigid body rotating about a fixed axis. Assume that external torques are absent. If $\mathcal{I}$ is the moment of inertia of the body relative to the axis of rotation, and φ is the angle of its rotation around the axis, then

$$L = \tfrac{1}{2}\mathcal{I}\dot{\varphi}^2, \quad p_\varphi = \mathcal{I}\dot{\varphi}, \quad H = \frac{p_\varphi^2}{2\mathcal{I}A}.$$

Find the action-angle variables for this system.

Problem 4. Consider an anharmonic oscillator described by the equation

$$\ddot{q} + \omega_0^2 \sin q = 0.$$

Examine the case where the energy satisfies $h > \omega_0^2$. Find the action variable and express the Hamiltonian in terms of the action variable. For convenience, introduce a new parameter

$$\kappa^2 = \frac{4\omega_0^2}{p_0^2}.$$