Problem 1. Show that the following transformation for the harmonic oscillator

$$Q = q(\cos\omega t + \omega t \sin\omega t) + \frac{p}{m\omega}(\omega t \cos\omega t - \sin\omega t)$$
$$P = m\omega q \sin\omega t + p\cos\omega t$$

is canonical transformation, and construct the generating function.

Problem 2. Let $H(q, p)$ be a time-independent Hamiltonian and let $F(q, p, t)$ be an integral of motion, i.e.

$$\frac{dF}{dt} \;=\; \frac{\partial F}{\partial t} + \{F, H\} \;=\; 0.$$

Show that $\frac{\partial F}{\partial t}$ and $\frac{\partial^2 F}{\partial t^2}$ are also integrals of motion.

Problem 3. Consider the one–dimensional harmonic oscillator with Hamiltonian

$$H(q, p) = \tfrac{1}{2}\,(p^2 + q^2).$$

Introduce the *holomorphic variables*

$$a = \frac{1}{\sqrt{2}}\,(p + iq), \qquad a^* = \frac{1}{\sqrt{2}}\,(p - iq).$$

(1) Compute the Poisson brackets $\{a, a\}$, $\{a^*, a^*\}$, and $\{a, a^*\}$ starting from $\{q, p\} = 1$. Decide whether the transformation $(q, p) \mapsto (a, a^*)$ is canonical.
(2) Express the Hamiltonian H in terms of the variables a and a^*.

Problem 4. The generating function be given by

$$F(q, Q) \;=\; \frac{1}{2\lambda}\sum_{k=1}^{n}\big(q_k - Q_k\big)^2,$$

where λ is a constant.

(1) Find the canonical transformation $(q, p) \mapsto (Q, P)$ generated by F. Write Q_k and P_k explicitly in terms of q_k and p_k.
(2) What transformation is obtained in the limit $\lambda \to 0$?