Problem 1. A physical system is described by the Lagrangian

$$L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) + a\,\rho^2\dot{\varphi},$$

where ρ, φ, z are cylindrical coordinates and a is a constant.

(a) Construct the Hamiltonian H of the system.
(b) Find three integrals of motion.
(c) Show that the radial equation of motion can be reduced to a quadrature of the form

$$t = \int \frac{d\rho}{\sqrt{\alpha - \dfrac{(\beta - a\rho^2)^2}{m^2\rho^2}}},$$

where α and β are constants.

Problem 2. Consider the one-dimensional harmonic oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

(a) Write down the Hamilton–Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0$$

for this system.
(b) Find a complete integral of the Hamilton–Jacobi equation of the form

$$S(q, \alpha, t),$$

where α is a constant parameter related to the energy.
(c) Solve the equation of motion via the generating function $S(q, \alpha, t)$.