Problem 1. Consider the Kepler problem with Hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2\mu} - \frac{\alpha}{r}, \qquad r = |\mathbf{r}|,\ \alpha > 0,$$

and angular momentum

$$\mathbf{M} = \mathbf{r} \times \mathbf{p}.$$

Define the Laplace–Runge–Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{M} - \mu\alpha\,\frac{\mathbf{r}}{r}.$$

(1) Use Hamilton's equations to show that the Laplace–Runge–Lenz vector is an integral of motion.
(2) Compute $\dot{\mathbf{A}}$ using

$$\dot{\mathbf{p}} = -\nabla U = -\frac{\alpha}{r^3}\mathbf{r}, \qquad \dot{\mathbf{r}} = \frac{\mathbf{p}}{\mu},$$

and show that every term cancels identically.

Problem 2. Consider the Hamilton–Jacobi equationfor the Kepler problem

$$\frac{1}{2\mu}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{\alpha}{r} = E.$$

(1) Separate variables in polar coordinates.
(2) Find ϕ in terms of r.