Problem 1. Find the Lagrangian function if the Hamiltonian function is given:

$$\text{(a)} \quad H = \frac{p_r^2}{2\theta^2} + \frac{p_\theta^2}{2r^2 \sin\theta} + r,$$

$$\text{(b)} \quad H = \ln p_x + p_x + \frac{p_y^2}{2},$$

$$\text{(c)} \quad H = \frac{t\, p_x^2}{2} + p_x p_y.$$

Problem 2. Let q and p be canonically conjugate variables. New coordinates Q, P are defined by the transformation

$$Q = \ln\bigl(1 + \sqrt{q}\cos p\bigr),$$
$$P = 2\bigl(1 + \sqrt{q}\cos p\bigr)\sqrt{q}\sin p.$$

Prove that this transformation is canonical.

Problem 3. The Lagrangian of the (open) Toda chain is given by

$$L(\{q_i\}, \{\dot{q}_i\}) = \sum_{i=1}^{N} \frac{1}{2}\dot{q}_i^2 - \sum_{i=1}^{N-1} e^{\,q_i - q_{i+1}}.$$

(1) Find the corresponding Hamiltonian $H(\{q_i\}, \{p_i\})$.
(2) Write the Hamilton–Jacobi equation for the principal function $S(\{q_i\}, t)$.

Problem 4. The Lagrangian function of the system is given by

$$L = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - U(x_1 - x_2) - U(x_1 - x_3) - U(x_2 - x_3),$$

where

$$U(x) = \frac{16}{x^4}.$$

Verify that the quantity

$$I = 2\,\dot{x}_1\dot{x}_2\dot{x}_3 - \dot{x}_1\,U(x_2 - x_3) - \dot{x}_2\,U(x_1 - x_3) - \dot{x}_3\,U(x_1 - x_2)$$

is an integral of motion.