Problem 1a. Find the frequency of small oscillations of the following system

$$L = \frac{\dot{x}^2}{x} - \frac{x}{\ln x}.$$

Problem 1b. Derive the expression for displacement for forced oscillations with the external force

$$F = a e^{-\gamma t} \cos(\beta t) \ ,$$

where a, γ and β are constants.

Problem 2a. Show that the eigenvectors of the symmetric matrix have the property that eigenvectors with different eigenvalues are orthogonal.

Problem 2b. Write the equations of motion for the symmetric top, i.e. for a rigid body with inertia I_3 around a symmetry axis, and I_1 around the two perpendicular principle axes of rotation.

Problem 3a. Derive the Rutherford's formula.

Problem 3b. Find under what conditions the following system is stable

$$\ddot{x} + \omega^2(t) x = 0 \ ,$$

where $\omega(t)$ is a periodic function.

Problem 4. Find the normal modes for the system given by the following Lagrangian

$$L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = \frac{1}{2} m l^2 (\dot{\phi}_1^2 + \dot{\phi}_2^2) - \frac{1}{2} m g l (\phi_1^2 + \phi_2^2) - \frac{k}{2} \left(\frac{l}{2}\right)^2 (\phi_2 - \phi_1)^2$$

Problem 5. Consider the following system with two degrees of freedom

$$\ddot{x} + \omega_x^2 x = 0$$
$$\ddot{y} + \omega_y^2 y = 0$$

Show that if $\omega_x = \omega_y$ and the phase difference is a multiple of π ($\varphi_x - \varphi_y = \pi n$) then the Lissajous curve is the diagonal of the rectangle $\{(x, y) : -A_x \leq x \leq A_x, \ -A_y \leq y \leq A_y\}$.